

TRANSITIVE 2-REPRESENTATIONS OF FINITARY 2-CATEGORIES

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ABSTRACT. In this article, we define and study the class of simple transitive 2-representations of finitary 2-categories. We prove a weak version of the classical Jordan-Hölder Theorem where the weak composition subquotients are given by simple transitive 2-representations. For a large class of finitary 2-categories we prove that simple transitive 2-representations are exhausted by cell 2-representations. Finally, we show that this large class contains finitary quotients of 2-Kac-Moody algebras.

1. INTRODUCTION

This article, for the first time, proves a general classification result for an axiomatically defined class of 2-representations of a large class of 2-categories covering most examples studied in the area of categorification.

More specifically, we study finitary 2-categories over an algebraically closed field which include the 2-category of Soergel bimodules associated to a finite Coxeter system (see [BG, So, EW]), an exhaustive family of quotients of 2-Kac-Moody algebras (see [BFK, KL, Ro1, CL, We]), quiver 2-categories constructed in [Xa] and the 2-category of projective functors on the module category of a finite dimensional algebra (see [MM1]). We define a new class of 2-representations for such 2-categories which we call *simple transitive 2-representations* and which we believe serves as the correct 2-analogue for the class of irreducible representations of an algebra. Our definition of simple transitive 2-representations comes in two layers, the first being a discrete transitive action of the multisemigroup of 1-morphisms (this alone is called *transitivity*), the second being the absence of categorical ideals in the representation invariant under the 2-action (this is what we refer to as *simplicity*).

For simple transitive 2-representations we obtain, for arbitrary finitary 2-categories, a weak version of the classical Jordan-Hölder Theorem, see Theorem 8, in which simple transitive 2-representations appear as weak composition subquotients of general finitary 2-representations. It turns out that any finitary 2-representation of a finitary 2-category has a filtration with subquotients being transitive 2-representations. In contrast to classical representation theory, transitive 2-representations do not seem to admit any natural filtration, however, they do have a well-defined simple top which is our weak composition subquotient. A different approach to the Jordan-Hölder theory for 2-Kac-Moody algebras is outlined in [Ro1, Subsection 5.1].

Our main result is Theorem 18 which provides a classification of simple transitive 2-representations for a large class of finitary 2-categories. The latter includes the 2-category of Soergel bimodules in type A , all of the above mentioned finitary quotients of 2-Kac-Moody algebras and the 2-category of projective functors on the

module category of a finite dimensional self-injective algebra. Moreover, it also includes all variations of the latter 2-category which constitute a list of finitary 2-categories from [MM3] satisfying a 2-analogue of simplicity for a finite dimensional algebra. The classification result states that for this class of 2-categories simple transitive 2-representations are precisely the cell 2-representations studied in [MM1, MM2, MM3]. In particular, this implies uniqueness of categorification of simple integrable modules for finite dimensional simple Lie algebras. The only comparable statement in the literature, for the 2-categorical analogue of $U(\mathfrak{sl}_2)$ and for a special class of 2-representations categorifying simple \mathfrak{sl}_2 -modules, was proved in [CR, Proposition 5.26].

The proof can be divided into two major parts. One of these (the proof of Theorem 18) reduces the problem to the case of the 2-category of projective functors on the module category of a finite dimensional self-injective algebra. The latter case is treated in Theorem 15 and relies on a detailed study of endomorphism algebras of certain bimodules and, crucially, on a classical result of Perron and Frobenius on the structure of real matrices with positive coefficients.

The article is organized as follows. In Section 2 we recall notions developed in [MM1, MM2, MM3] and state the Perron-Frobenius Theorem. In Section 3 we introduce transitive and simple transitive 2-representations and gather examples and preliminary results. Section 4 presents the statement and proof of our weak Jordan-Hölder Theorem. Section 5 is devoted to the proof of our main result in the case of the 2-category of projective functors on the module category of a finite dimensional self-injective algebra. Section 6 establishes the main result in the general case. Finally, in Section 7 we provide and study examples, including our family of quotients of 2-Kac-Moody algebras.

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2. PRELIMINARIES

2.1. Notation. Throughout, we let \mathbb{k} denote an algebraically closed field.

A *2-category* is a category enriched over the category of small categories. A 2-category \mathcal{C} consists of objects (denoted $\mathbf{i}, \mathbf{j}, \mathbf{k}, \dots$), 1-morphisms (denoted F, G, H, \dots) and 2-morphisms (denoted $\alpha, \beta, \gamma, \dots$). For $\mathbf{i} \in \mathcal{C}$, the identity 1-morphism is denoted $1_{\mathbf{i}}$ and, for a 1-morphism F , the corresponding identity 2-morphism is denoted id_F . Composition of 1-morphisms is denoted by \circ , horizontal composition of 2-morphisms is denoted by \circ_0 and vertical composition of 2-morphisms is denoted by \circ_1 . We let \mathbf{Cat} denote the 2-category of small categories.

2.2. Finitary 2-categories. An additive \mathbb{k} -linear category is called *finitary* if it is idempotent split, has finitely many isomorphism classes of indecomposable objects and finite dimensional \mathbb{k} -vector spaces of morphisms. Denote by $\mathfrak{A}_{\mathbb{k}}^f$ the 2-category whose objects are finitary additive \mathbb{k} -linear categories, 1-morphisms are additive \mathbb{k} -linear functors and 2-morphisms are natural transformations of functors.

A *finitary* 2-category (over \mathbb{k}) is a 2-category \mathcal{C} with the following properties:

- it has a finite number of objects;
- for any pair \mathbf{i}, \mathbf{j} of objects in \mathcal{C} , the category $\mathcal{C}(\mathbf{i}, \mathbf{j})$ is in $\mathfrak{A}_{\mathbb{k}}^f$ and horizontal composition is both additive and \mathbb{k} -linear;
- for any $\mathbf{i} \in \mathcal{C}$, the 1-morphism $\mathbb{1}_{\mathbf{i}}$ is indecomposable.

We refer to [Le, McL] for more general details on abstract 2-categories and to [MM1, MM2, MM3, MM4] for more information on finitary 2-categories.

2.3. 2-representations. Let \mathcal{C} be a finitary 2-category. By a *2-representation* of \mathcal{C} we mean a strict 2-functor from \mathcal{C} to \mathbf{Cat} . By a *finitary 2-representation* of \mathcal{C} we mean a strict 2-functor from \mathcal{C} to $\mathfrak{A}_{\mathbb{k}}^f$. Our 2-representations are generally denoted by $\mathbf{M}, \mathbf{N}, \dots$ with one exception: for $\mathbf{i} \in \mathcal{C}$ we have the *principal* 2-representation $\mathbb{P}_{\mathbf{i}} := \mathcal{C}(\mathbf{i}, -)$. Finitary 2-representations of \mathcal{C} form a 2-category, denoted $\mathcal{C}\text{-afmod}$, whose 1-morphisms are 2-natural transformations and whose 2-morphisms are modifications (see [Le, MM3]).

Two 2-representations \mathbf{M} and \mathbf{N} of \mathcal{C} are called *equivalent* if there exists a 2-natural transformation $\Phi : \mathbf{M} \rightarrow \mathbf{N}$ such that $\Phi_{\mathbf{i}}$ is an equivalence for each \mathbf{i} .

Let \mathbf{M} be a 2-representation of \mathcal{C} . Assume that $\mathbf{M}(\mathbf{i})$ is an idempotent split additive category for each $\mathbf{i} \in \mathcal{C}$. For any collection of objects $X_i \in \mathbf{M}(\mathbf{i}_i)$, where $i \in I$, the additive closure of all objects of the form $\mathbf{F}X_i$, where $i \in I$ and \mathbf{F} runs through all 1-morphisms of \mathcal{C} is stable under the action of \mathcal{C} and hence inherits the structure of a 2-representation by restriction. This 2-representation will be denoted $\mathbf{G}_{\mathbf{M}}(\{X_i : i \in I\})$.

To simplify notation, we will often write $\mathbf{F}X$ for $\mathbf{M}(\mathbf{F})X$ where \mathbf{F} is a 1-morphism.

2.4. Combinatorics of finitary 2-categories. Let \mathcal{C} be a finitary 2-category. Denote by $\mathcal{S}(\mathcal{C})$ the multisemigroup of isomorphism classes of 1-morphisms in \mathcal{C} , see [MM2, Section 3]. As usual, we define the left preorder \geq_L on $\mathcal{S}(\mathcal{C})$ as follows: for two 1-morphisms \mathbf{F}, \mathbf{G} we set $\mathbf{G} \geq_L \mathbf{F}$ provided that there is a 1-morphism \mathbf{H} such that \mathbf{G} is isomorphic to a direct summand of $\mathbf{H} \circ \mathbf{F}$. Equivalence classes for \geq_L are called *left cells*. Right and two-sided preorders \geq_R and \geq_J and respective cells are defined analogously.

2.5. Weakly fiat and fiat 2-categories. For a 2-category \mathcal{C} there are three ways of creating an opposite 2-category.

- We can reverse both 1- and 2-morphisms.
- We can reverse only 1-morphisms.

- We can reverse only 2-morphisms.

In the present paper we let \mathcal{C}^{op} denote the first of the three choices above.

A finitary 2-category \mathcal{C} is called *weakly fiat* provided that

- there is a weak equivalence $*$: $\mathcal{C} \rightarrow \mathcal{C}^{\text{op}}$;
- for any pair $\mathbf{i}, \mathbf{j} \in \mathcal{C}$ and every 1-morphism $F \in \mathcal{C}(\mathbf{i}, \mathbf{j})$ we have 2-morphisms $\alpha : F \circ F^* \rightarrow \mathbb{1}_{\mathbf{j}}$ and $\beta : \mathbb{1}_{\mathbf{i}} \rightarrow F^* \circ F$ such that $\alpha_F \circ_1 F(\beta) = \text{id}_F$ and $F^*(\alpha) \circ_1 \beta_{F^*} = \text{id}_{F^*}$.

If $*$ is involutive, then \mathcal{C} is called *fiat*, see [MM1, MM2].

2.6. 2-ideals. Let \mathcal{C} be a 2-category. A *left 2-ideal* \mathcal{I} of \mathcal{C} consists of the same objects as \mathcal{C} and for each pair \mathbf{i}, \mathbf{j} of objects an ideal $\mathcal{I}(\mathbf{i}, \mathbf{j})$ in $\mathcal{C}(\mathbf{i}, \mathbf{j})$ such that \mathcal{I} is stable under the left horizontal multiplication with 1- and 2-morphisms in \mathcal{C} . Similarly one defines *right 2-ideals* and *two-sided 2-ideals*. The latter will simply be called *2-ideals*. For example, each principal 2-representation can be interpreted as a left 2-ideal in \mathcal{C} .

Let \mathcal{C} be a 2-category and \mathbf{M} be a 2-representation of \mathcal{C} . An *ideal* \mathbf{I} of \mathbf{M} is a collection of ideals $\mathbf{I}(\mathbf{i})$ in $\mathbf{M}(\mathbf{i})$ for each $\mathbf{i} \in \mathcal{C}$ stable under the action of \mathcal{C} in the following sense: for any morphism $\eta \in \mathbf{I}$ and any 1-morphism F the composition $\mathbf{M}(F)(\eta)$ (if it is defined) is in \mathbf{I} . For example, left 2-ideals of \mathcal{C} give rise to ideals in principal 2-representations.

2.7. Abelianization. Let \mathcal{A} be a finitary additive \mathbb{k} -linear category. Then the *abelianization* $\overline{\mathcal{A}}$ of \mathcal{A} is the category whose objects are diagrams $X \xrightarrow{\eta} Y$ where $X, Y \in \mathcal{A}$ and $\eta \in \mathcal{A}(X, Y)$ and morphisms are equivalence classes of solid commutative diagrams of the form

$$\begin{array}{ccc} X & \xrightarrow{\eta} & Y \\ \tau_1 \downarrow & \swarrow \tau_3 & \downarrow \tau_2 \\ X' & \xrightarrow{\eta'} & Y' \end{array}$$

modulo the subspace spanned by those diagrams for which there exists τ_3 as indicated by the dashed arrow such that $\eta' \tau_3 = \tau_2$. The category $\overline{\mathcal{A}}$ is abelian (cf. [Fr]) and is equivalent to the category of modules over the finite dimensional \mathbb{k} -algebra

$$\text{End}_{\mathcal{A}}(P)^{\text{op}} \quad \text{where} \quad P := \bigoplus_{Q \in \text{Ind}(\mathcal{A})/\cong} Q.$$

Let \mathcal{C} be a 2-category and \mathbf{M} a finitary 2-representation of \mathcal{C} . Then the *abelianization* of \mathbf{M} is the 2-representation $\overline{\mathbf{M}}$ of \mathcal{C} which assigns to each $\mathbf{i} \in \mathcal{C}$ the category $\overline{\mathbf{M}}(\mathbf{i})$ with the action of \mathcal{C} defined on diagrams component-wise.

Directly from the definition it follows that the action of each 1-morphism on the abelianization of any finitary 2-representation is right exact.

A finitary 2-representation \mathbf{M} of \mathcal{C} will be called *exact* provided that $\overline{\mathbf{M}}(\mathbf{F})$ is exact for any 1-morphism \mathbf{F} in \mathcal{C} . For example, any finitary 2-representation of a weakly fiat 2-category is exact.

2.8. Perron-Frobenius Theorem. We will use the following classical result due to Perron and Frobenius, see the original papers [Fro1, Fro2, Pe] or the detailed exposition in [Me, Chapter 8].

Theorem 1. *Let $A = (a_{i,j})$ be a real $n \times n$ matrix with strictly positive coefficients.*

- (i) *A has a positive real eigenvalue, call it r , such that any other (possibly complex) eigenvalue of A has a strictly smaller absolute value.*
- (ii) *The eigenvalue r appears with multiplicity one in the characteristic polynomial of A .*
- (iii) *There exists a real eigenvector, call it \mathbf{v} , for eigenvalue r with strictly positive coefficients, moreover, any real eigenvector of A with strictly positive coefficients is a multiple of \mathbf{v} .*
- (iv) *The eigenvalue r satisfies*

$$\min_j \left\{ \sum_i a_{ij} \right\} \leq r \leq \max_j \left\{ \sum_i a_{ij} \right\}.$$

Corollary 2. *Assume that A is as in Theorem 1 and has rank one. Then, if either inequality in Theorem 1(iv) is an equality, then both inequalities are equalities and all columns of A coincide.*

Proof. If A has rank one, then all columns of A are proportional to \mathbf{v} and the trace of A equals r . Assume, for example, that $\min_j \{ \sum_i a_{ij} \} = \sum_i a_{i1} = r$. Set $\lambda_1 = 1$ and for $j = 2, 3, \dots, n$ let λ_j be the positive real number (≥ 1) such that the j -th column equals λ_j times the first column. Then, we have

$$\sum_i a_{i1} = r = \text{trace}(A) = \sum_i a_{ii} = \sum_i \lambda_i a_{i1} \geq \sum_i a_{i1} = r.$$

It follows that $\lambda_j = 1$ for all j . The case where the second inequality is an equality is similar. \square

3. TRANSITIVE 2-REPRESENTATIONS

In this section, \mathcal{C} will be a finitary 2-category.

3.1. Definition. Let \mathbf{M} be a finitary 2-representations of \mathcal{C} . We will say that \mathbf{M} is *transitive* provided that for every \mathbf{i} and for every non-zero object $X \in \mathbf{M}(\mathbf{i})$ we have $\mathbf{G}_{\mathbf{M}}(\{X\}) = \mathbf{M}$.

3.2. Example: transitive group actions. Let $G = (G, \cdot)$ be a finite group. Consider the finitary 2-category $\mathcal{G} = \mathcal{G}_G$ defined as follows:

- \mathcal{G} has one object \clubsuit ;

- 1-morphisms in \mathcal{G} are $\bigoplus_{g \in G} F_g^{\oplus k_g}$ where all $k_g \geq 0$;
- composition of 1-morphisms is given by $F_g \circ F_h = F_{gh}$ and extended by biadditivity;
- non-zero 2-morphisms between indecomposable 1-morphisms are just scalar multiples of the identity, 2-morphisms between decomposable 1-morphisms are matrices of morphisms between the corresponding indecomposable summands;
- vertical composition of 2-morphisms is given by matrix multiplication;
- horizontal composition of 2-morphisms is given by tensor product of matrices.

The 2-category \mathcal{G} is finitary by definition. Moreover, it is even a fiat 2-category (where $*$ is induced by $g \mapsto g^{-1}$).

Let H be a subgroup of G . Let \mathcal{A} be a small category equivalent to $\mathbb{k}\text{-mod}$. Consider the category

$$\mathcal{G}_{H,\mathcal{A}} := \bigoplus_{gH \in G/H} \mathcal{A}_{(gH)},$$

where (gH) is a formal index. Now define the 2-representation $\mathbf{M}_{H,\mathcal{A}}$ of \mathcal{G}

- on the object by $\mathbf{M}_{H,\mathcal{A}}(\clubsuit) = \mathcal{G}_{H,\mathcal{A}}$;
- on 1-morphisms by $\mathbf{M}_{H,\mathcal{A}}(F_g) = (\varphi_{xH,yH})_{xH,yH \in G/H}$ where

$$\varphi_{xH,yH} = \begin{cases} \text{Id}_{\mathcal{A}}, & gyH = xH; \\ 0, & \text{otherwise;} \end{cases}$$

- on 2-morphisms $\mathbf{M}_{H,\mathcal{A}}$ in the obvious way using scalar multiples of the identity natural transformations.

It follows from the definition that $\mathbf{M}_{H,\mathcal{A}}$ is a transitive 2-representation of \mathcal{G} . This 2-representation categorifies the classical transitive action of G on G/H .

Note that in the above construction instead of \mathcal{A} we can take any small finitary additive \mathbb{k} -linear category \mathcal{B} with one isomorphism class of indecomposable objects.

This example generalizes, in the obvious way, to finite semigroups. One major difference is that in the latter case the 2-category obtained will not be fiat but only finitary. Another difference is that while any transitive action of a finite group on a finite set is equivalent to the action on some G/H , transitive actions of semigroups are more complicated, see e.g. [GM, Chapter 10].

3.3. Cell 2-representations. Here we use the approach from [MM2] to construct cell 2-representations for arbitrary finitary 2-categories.

Let \mathcal{L} be a left cell in \mathcal{C} . Then there is $\mathbf{i} = \mathbf{i}_{\mathcal{L}} \in \mathcal{C}$ such that every 1-morphism in \mathcal{L} has domain \mathbf{i} . Consider the principal 2-representation $\mathbb{P}_{\mathbf{i}}$. For $\mathbf{j} \in \mathcal{C}$ let

$\mathbf{N}(j)$ denote the additive closure in $\mathbb{P}_i(j)$ of all 1-morphisms $F \in \mathcal{C}(i, j)$ such that $F \geq_L \mathcal{L}$. Then \mathbf{N} is a 2-subrepresentation of \mathbb{P}_i .

Lemma 3. *There is a unique maximal ideal \mathbf{I} in \mathbf{N} which does not contain id_F for any $F \in \mathcal{L}$.*

Proof. Being an ideal of an additive category, \mathbf{I} is uniquely determined by its morphisms between indecomposable objects. If $F \in \mathcal{L} \cap \mathcal{C}(i, j)$, then the algebra of 2-endomorphisms of F is local as F is indecomposable. Therefore the part of $\text{End}_{\mathcal{C}(i, j)}(F)$ contained in \mathbf{I} belongs to the radical of $\text{End}_{\mathcal{C}(i, j)}(F)$. As the sum of two subspaces of the radical is contained in the radical, we conclude that the sum of all left ideals in \mathbf{N} which do not contain id_F for any $F \in \mathcal{L}$ still has the latter property. The claim follows. \square

The quotient 2-functor $\mathbf{C}_{\mathcal{L}} := \mathbf{N}/\mathbf{I}$, where \mathbf{I} is given by Lemma 3, is called the *(additive) cell 2-representation* of \mathcal{C} associated to \mathcal{L} . From the definitions, it follows directly that $\mathbf{C}_{\mathcal{L}}$ is a transitive 2-representation of \mathcal{C} .

3.4. A more exotic example. Similarly to Subsection 3.2 one defines a 2-category \mathcal{C} with one object, indecomposable 1-morphisms $\mathbb{1}$ and F , with the multiplication table

$$\begin{array}{c|c|c} \circ & \mathbb{1} & F \\ \hline \mathbb{1} & \mathbb{1} & F \\ \hline F & F & F \oplus F \end{array}$$

and only scalar multiples of the identity 2-morphisms for indecomposable 1-morphisms. This 2-category \mathcal{C} has two left cells (corresponding to the two indecomposable 1-morphisms), so we have the respective cell 2-representations. These are transitive, see Subsection 3.3. Similarly to Subsection 3.2 one can construct a rather different transitive 2-representation on a category $\mathcal{A} \oplus \mathcal{A}$, where \mathcal{A} is as in Subsection 3.2, by mapping the 1-morphism F to the functor

$$\begin{pmatrix} \text{Id}_{\mathcal{A}} & \text{Id}_{\mathcal{A}} \\ \text{Id}_{\mathcal{A}} & \text{Id}_{\mathcal{A}} \end{pmatrix}.$$

3.5. Simple transitive 2-representations. Let \mathbf{M} be a transitive 2-representation of \mathcal{C} .

Lemma 4. *There is a unique maximal ideal \mathbf{I} in \mathbf{M} which does not contain any identity morphisms apart from the one for the zero object.*

Proof. Mutatis mutandis proof of Lemma 3. \square

The main idea of the following definition generalizes [MM2, Subsection 6.5]. A transitive 2-representation \mathbf{M} of \mathcal{C} is called *simple transitive* provided that its unique maximal ideal given by Lemma 4 is the zero ideal. For a transitive 2-representation \mathbf{M} denote by $\underline{\mathbf{M}}$ the quotient of \mathbf{M} by the ideal \mathbf{I} given by Lemma 4. We will loosely call $\underline{\mathbf{M}}$ the *simple transitive quotient* of \mathbf{M} .

3.6. Examples of simple transitive 2-representations. Lemma 3 implies that each cell 2-representation of \mathcal{C} is simple transitive. Furthermore, transitive 2-representations $\mathbf{M}_{H,\mathcal{A}}$ of \mathcal{G} constructed in Subsection 3.2 are simple transitive (and these are not equivalent to cell 2-representations in general). In fact, the next proposition shows that these exhaust all simple transitive 2-representations of \mathcal{G} .

Proposition 5. *Every simple transitive 2-representations of \mathcal{G} is equivalent to $\mathbf{M}_{H,\mathcal{A}}$ for some subgroup H of G and a skeletal category \mathcal{A} equivalent to $\mathbb{k}\text{-mod}$.*

Proof. Let \mathbf{M} be a simple transitive 2-representation of \mathcal{G} . Invertibility of each F_g implies that F_g sends non-isomorphic objects to non-isomorphic objects, indecomposable objects to indecomposable objects and radical morphisms to radical morphisms. Therefore the ideal \mathbf{I} given by Lemma 4 coincides with the radical of $\mathbf{M}(\clubsuit)$. By simple transitivity, we hence obtain that the radical of $\mathbf{M}(\clubsuit)$ is zero and thus $\mathbf{M}(\clubsuit)$ is a semi-simple category.

As each F_g sends indecomposable objects to indecomposable objects, G induces a transitive action on the set of isomorphism classes of indecomposable objects in $\mathbf{M}(\clubsuit)$. Fix an indecomposable object $X \in \mathbf{M}(\clubsuit)$ and set

$$H := \{h \in G : F_h X \cong X\}.$$

Let \mathcal{A} be a skeletal category equivalent to $\mathbb{k}\text{-mod}$. Consider the (unique!) functor $\Phi : \mathbf{M}(\clubsuit) \rightarrow \mathcal{G}_H$ which sends an indecomposable object $Y \cong F_g X$ for some $g \in G$ to the unique indecomposable object in $\mathcal{A}_{(gH)}$. Then Φ is easily checked to give an equivalence between \mathbf{M} and $\mathbf{M}_{H,\mathcal{A}}$. The claim follows. \square

Note that Proposition 5 does not extend to all transitive 2-representations in an obvious way. For example, let G be the cyclic group of order two. Then G acts by automorphisms on the finite dimensional \mathbb{k} -algebra A given by the quiver

$$\begin{array}{ccc} & a & \\ 1 & \xrightarrow{\quad} & 2 \\ & b & \end{array}$$

with relations $ab = ba = 0$ (the non-trivial automorphism is given by the automorphism of the quiver which swaps 1 with 2 and a with b). This induces a transitive action of G and hence of the corresponding 2-category \mathcal{G} on any skeletal category equivalent to the category of finite dimensional projective A -modules. We refer to [AM, Section 2] for more details.

3.7. Strongly simple 2-representations are (simple) transitive. In parallel to [MM1, Subsection 6.2], we call a finitary 2-representation \mathbf{M} of \mathcal{C} *strongly simple* provided that for any $\mathbf{i}, \mathbf{j} \in \mathcal{C}$ with $\overline{\mathbf{M}}(\mathbf{i})$ nonzero, any simple object $L \in \overline{\mathbf{M}}(\mathbf{i})$ and any pair P, Q of indecomposable projectives in $\overline{\mathbf{M}}(\mathbf{j})$, there exist indecomposable 1-morphisms F and G such that $FL \cong P$, $GL \cong Q$ and the evaluation map $\text{Hom}_{\mathcal{C}(\mathbf{i},\mathbf{j})}(F, G) \rightarrow \text{Hom}_{\overline{\mathbf{M}}(\mathbf{j})}(FL, GL)$ is surjective.

Proposition 6. *Let \mathcal{C} be a finitary 2-category and \mathbf{M} a strongly simple finitary 2-representation of \mathcal{C} .*

- (i) *The 2-representation \mathbf{M} is transitive.*
- (ii) *If \mathbf{M} is exact (in particular, if \mathcal{C} is weakly fiat), then \mathbf{M} is simple transitive.*

Proof. Let X be a non-zero indecomposable object in some $\mathbf{M}(\mathbf{i})$ and L be its simple top in $\overline{\mathbf{M}}(\mathbf{i})$. Let Y be a non-zero indecomposable object in some $\mathbf{M}(\mathbf{j})$. By definition of strong simplicity, there is an indecomposable 1-morphism F such that $FL \cong Y$. This means that Y is isomorphic to a direct summand of FX and hence \mathbf{M} is transitive. This proves claim (i).

Let $X, Y \in \overline{\mathbf{M}}(\mathbf{i})$ be two indecomposable projective objects and $\eta : X \rightarrow Y$ be a non-zero morphism. Denote by $L \in \overline{\mathbf{M}}(\mathbf{i})$ the simple top of X . Choose two 1-morphisms F and G in \mathcal{C} such that $FL \cong X$ and $GL \cong Y$. Consider a finite dimensional \mathbb{k} -algebra B such that $\overline{\mathbf{M}}(\mathbf{i}) \cong B\text{-mod}$. For simplicity, we identify $\overline{\mathbf{M}}(\mathbf{i})$ and $B\text{-mod}$. Let e, e' be two primitive idempotents of B such that $X \cong Be$ and $Y \cong Be'$. Then, by Lemma 13, the functor $\overline{\mathbf{M}}(F)$ surjects onto the projective functor $Be \otimes_{\mathbb{k}} eB \otimes_B -$. Similarly, the functor $\overline{\mathbf{M}}(G)$ surjects onto the projective functor $Be' \otimes_{\mathbb{k}} eB \otimes_B -$.

Now, for any non-zero map $\eta' : Be \rightarrow Be'$ the induced map

$$\text{Id}_{Be} \otimes \text{Id}_{eB} \otimes \eta' : Be \otimes_{\mathbb{k}} eB \otimes_B Be \rightarrow Be \otimes_{\mathbb{k}} eB \otimes_B Be'$$

contains, as a direct summand, the identity map on Be . This implies that the ideal \mathbf{I} in \mathbf{M} generated by η contains the identity morphism on X . Therefore \mathbf{M} is simple transitive. \square

Example 7. The claim of Proposition 6(ii) fails for general finitary 2-representations. Consider the algebra $D = \mathbb{k}[x]/(x^2)$ of dual numbers. Let \mathcal{A} be a small category equivalent to $D\text{-mod}$ and \mathcal{C} the finitary category with one object \clubsuit which we identify with \mathcal{A} , with indecomposable 1-morphisms being endofunctors of \mathcal{A} isomorphic to either the identity functor or tensoring with the D - D -bimodule $D \otimes_{\mathbb{k}} \mathbb{k}$, and 2-morphisms being natural transformations of functors. Then the defining 2-representation of \mathcal{C} , i.e. the natural 2-action of \mathcal{C} on \mathcal{A} , is clearly strongly simple. However, as tensoring with $D \otimes_{\mathbb{k}} \mathbb{k}$ annihilates the non-zero nilpotent endomorphism of ${}_D D$, this 2-representation is not simple transitive.

Note also that the example of a transitive 2-representation considered in Subsection 3.4 is, clearly, simple transitive but not strongly simple.

4. WEAK JORDAN-HÖLDER THEORY

In this section, \mathcal{C} will be a finitary 2-category.

4.1. The action preorder. Let \mathbf{M} be a finitary 2-representation of \mathcal{C} . Consider the (finite) set $\text{Ind}(\mathbf{M})$ of isomorphism classes of indecomposable objects in all $\mathbf{M}(\mathbf{i})$ where $\mathbf{i} \in \mathcal{C}$. For $X, Y \in \text{Ind}(\mathbf{M})$ set $X \geq Y$ provided that there is a 1-morphism F in \mathcal{C} such that X is isomorphic to a direct summand of FY . Clearly, \geq is a partial preorder on $\text{Ind}(\mathbf{M})$ which we will call the *action preorder*.

Let \sim be the equivalence relation defined by $X \sim Y$ if and only if $X \geq Y$ and $Y \geq X$. Note that \mathbf{M} is transitive if and only if we have exactly one equivalence class, namely the whole of $\text{Ind}(\mathbf{M})$. The preorder \geq induces a genuine partial order on the set $\text{Ind}(\mathbf{M})/\sim$ which, abusing notation, we will denote by the same symbol.

4.2. 2-subrepresentations and subquotients associated to coideals. Let Q be a coideal in $\text{Ind}(\mathbf{M})/\sim$. For $\mathbf{i} \in \mathcal{C}$ consider the additive closure $\mathbf{M}_Q(\mathbf{i})$ in $\mathbf{M}(\mathbf{i})$ of all indecomposable objects $X \in \mathbf{M}(\mathbf{i})$ whose equivalence class belongs to Q . Then \mathbf{M}_Q has the natural structure of a 2-representations of \mathcal{C} given by restriction from \mathbf{M} . This is the 2-subrepresentation of \mathbf{M} associated to Q .

Suppose we are given a pair Q, R of coideals in $\text{Ind}(\mathbf{M})/\sim$ such that $Q \subset R$. For $\mathbf{i} \in \mathcal{C}$ let $\mathbf{I}(\mathbf{i})$ denote the ideal in $\mathbf{M}_R(\mathbf{i})$ generated by the identities on the objects in $\mathbf{M}_Q(\mathbf{i})$. Then we can form the quotient category $\mathbf{M}_{R/Q}(\mathbf{i}) := \mathbf{M}_R(\mathbf{i})/\mathbf{I}(\mathbf{i})$ and the 2-functor \mathbf{M}_R induces the 2-functor $\mathbf{M}_{R/Q}$ which sends \mathbf{i} to $\mathbf{M}_{R/Q}(\mathbf{i})$. This is the 2-subquotient of \mathbf{M} associated to $Q \subset R$. Note that $|R \setminus Q| = 1$ implies that the 2-representation $\mathbf{M}_{R/Q}$ is transitive.

For $r \in \text{Ind}(\mathbf{M})/\sim$ let X_r be the maximal coideal in $\text{Ind}(\mathbf{M})/\sim$ which does not contain r . Then r becomes the minimum element in $(\text{Ind}(\mathbf{M})/\sim) \setminus X_r$ with respect to the induced order. Let $Y_r := X_r \cup \{r\}$. Then Y_r is a coideal in $\text{Ind}(\mathbf{M})/\sim$. Therefore we have the associated quotient \mathbf{M}_{Y_r/X_r} and we set $\underline{\mathbf{M}}_r := \mathbf{M}_{Y_r/X_r}$.

4.3. Weak Jordan-Hölder series. Consider a filtration

$$\mathcal{Q} : \quad \emptyset = Q_0 \subsetneq Q_1 \subsetneq Q_2 \subsetneq \cdots \subsetneq Q_k = \text{Ind}(\mathbf{M})/\sim$$

of coideals such that $|Q_i \setminus Q_{i-1}| = 1$ for all i . Such a filtration will be called a *complete filtration*. With such a filtration we associate a filtration of 2-subrepresentations

$$(1) \quad 0 \subset \mathbf{M}_{Q_1} \subset \mathbf{M}_{Q_2} \subset \cdots \subset \mathbf{M}_{Q_k} = \mathbf{M}$$

and the corresponding sequence

$$(2) \quad \underline{\mathbf{M}}_{Q_1}, \underline{\mathbf{M}}_{Q_2/Q_1}, \underline{\mathbf{M}}_{Q_3/Q_2}, \dots, \underline{\mathbf{M}}_{Q_k/Q_{k-1}}$$

of *simple transitive subquotients*. The filtration (1) is called a *weak Jordan-Hölder series* of \mathbf{M} and the elements in (2) are also called *weak composition subquotients*.

4.4. Weak Jordan-Hölder theorem. The main result of this section is the following weak version of the classical Jordan-Hölder theorem.

Theorem 8. *Let \mathcal{C} be a finitary 2-category and \mathbf{M} a finitary 2-representation of \mathcal{C} . Let further \mathcal{Q} and \mathcal{R} be two complete filtrations of $\text{Ind}(\mathbf{M})/\sim$. Let $\mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_k$ be the sequence of simple transitive subquotients associated to \mathcal{Q} and $\mathbf{L}'_1, \mathbf{L}'_2, \dots, \mathbf{L}'_l$ be the sequence of simple transitive subquotients associated to \mathcal{R} . Then $k = l$ and there is a bijection $\sigma : \{1, 2, \dots, k\} \rightarrow \{1, 2, \dots, k\}$ such that \mathbf{L}_i and $\mathbf{L}'_{\sigma(i)}$ are equivalent for all $i \in \{1, 2, \dots, k\}$.*

Proof. Note first that we have $k = l = |\text{Ind}(\mathbf{M})/\sim|$ by definition. Let $r \in \text{Ind}(\mathbf{M})/\sim$. Then there are unique $i, j \in \{1, 2, \dots, k\}$ such that $r = Q_i \setminus Q_{i-1}$ and $r = R_j \setminus R_{j-1}$. To prove the assertion it is enough to show that the 2-representations $\underline{\mathbf{M}}_r$, \mathbf{L}_i and \mathbf{L}'_j are equivalent. By symmetry, it is enough to show that $\underline{\mathbf{M}}_r$ and \mathbf{L}_i are equivalent.

Let \mathbf{I} be the ideal in \mathbf{M}_{Y_r} used to define \mathbf{M}_{Y_r/X_r} . Similarly, let \mathbf{J} be the ideal in \mathbf{M}_{Q_i} used to define $\mathbf{M}_{Q_i/Q_{i-1}}$. By construction of X_r , we have $Q_{i-1} \subset X_r$ and hence also $Q_i \subset Y_r$. The inclusion $Q_i \subset Y_r$ induces a faithful 2-natural transformation from \mathbf{M}_{Q_i} to \mathbf{M}_{Y_r} which gives, by taking the quotient, a strong transformation from \mathbf{M}_{Q_i}

to \mathbf{M}_{Y_r/X_r} . Since $Q_{i-1} \subset X_r$, for any indecomposable objects M and N whose \sim -classes belong to r , we have $\mathbf{J}(M, N) \subset \mathbf{I}(M, N)$. Therefore the strong transformation from \mathbf{M}_{Q_i} to \mathbf{M}_{Y_r/X_r} factors through $\mathbf{M}_{Q_i/Q_{i-1}}$. This gives a 2-natural transformation from $\mathbf{M}_{Q_i/Q_{i-1}}$ to \mathbf{M}_{Y_r/X_r} which is surjective on morphisms. Note that both 2-representations $\mathbf{M}_{Q_i/Q_{i-1}}$ and \mathbf{M}_{Y_r/X_r} are transitive. Taking now the quotient by the unique maximal ideal given by Lemma 4 induces an equivalence between the corresponding simple transitive quotients, that is between \mathbf{L}_i and $\underline{\mathbf{M}}_r$. The claim follows. \square

4.5. Example: weak composition subquotients for principal 2-representations. Consider the principal 2-representation \mathbb{P}_i for $i \in \mathcal{C}$. The action preorder \geq for \mathbb{P}_i coincides with the restriction to \mathbb{P}_i of the preorder \geq_L . Therefore $\text{Ind}(\mathbb{P}_i)$ coincides with the set of isomorphism classes of 1-morphisms in \mathcal{C} with domain i . The set $\text{Ind}(\mathbb{P}_i)/\sim$ thus becomes the set of all left cells with domain i . Comparing Subsection 3.3 with Subsection 4.3, we see that weak composition subquotients of \mathbb{P}_i are exactly the cell 2-representations for left cells with domain i .

5. CLASSIFICATION OF TRANSITIVE 2-REPRESENTATIONS FOR \mathcal{C}_A

5.1. The 2-category \mathcal{C}_A . Let A be a basic self-injective connected \mathbb{k} -algebra of finite dimension m . Fix a small category \mathcal{A} equivalent to $A\text{-mod}$. We assume that \mathcal{A} is not semi-simple. Define the 2-category \mathcal{C}_A as follows (cf. [MM1, Subsection 7.3]):

- \mathcal{C}_A has one object \clubsuit (which we identify with \mathcal{A});
- 1-morphisms in \mathcal{C}_A are direct sums of functors with summands isomorphic to the identity functor or to tensoring with projective A - A -bimodules;
- 2-morphisms in \mathcal{C}_A are natural transformations of functors.

Functors isomorphic to tensoring with projective A - A -bimodules will be called *projective functors*.

Fix some decomposition $1 = e_1 + e_2 + \dots + e_n$ of the identity in A into a sum of primitive orthogonal idempotents. The 2-category \mathcal{C}_A has a unique minimal two-sided cell consisting of the isomorphism class of the identity morphism. It has one other two-sided cell \mathcal{J} consisting of the isomorphism classes of functors F_{ij} given by tensoring with the indecomposable bimodules $Ae_i \otimes e_j A$, where $i, j \in \{1, 2, \dots, n\}$. Left and right cells in \mathcal{J} are

$$\mathcal{L}_j := \{F_{ij} : i \in \{1, 2, \dots, n\}\} \quad \text{and} \quad \mathcal{R}_i := \{F_{ij} : j \in \{1, 2, \dots, n\}\},$$

where $i, j \in \{1, 2, \dots, n\}$. We have

$$F_{ij} \circ F_{st} \cong F_{it}^{\oplus \dim(e_j A e_s)}.$$

Let $\sigma : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ be the *Nakayama* bijection given by requiring $\text{soc } Ae_i \cong \text{top } Ae_{\sigma(i)}$ which is equivalent to $Ae_i \cong \text{Hom}_{\mathbb{k}}(e_{\sigma(i)} A, \mathbb{k})$. Since

$$\text{Hom}_A(Ae_i \otimes_{\mathbb{k}} e_j A, -) \cong \text{Hom}_{\mathbb{k}}(e_j A, \mathbb{k}) \otimes_{\mathbb{k}} e_i A \otimes_A -,$$

see e.g. [MM1, Subsection 7.3], we have that $(F_{ij}, F_{\sigma^{-1}(j)i})$ is an adjoint pair of functors. This implies that \mathcal{C}_A is weakly fiat with $*$ defined on 1-morphisms by $F_{ij}^* = F_{\sigma^{-1}(j)i}$.

We set $F := \bigoplus_{i,j=1}^n F_{ij}$. Since A is basic and

$$A \otimes_{\mathbf{k}} A \otimes_A A \otimes_{\mathbf{k}} A \cong A \otimes_{\mathbf{k}} A^{\oplus m},$$

we have

$$(3) \quad F \circ F \cong F^{\oplus m}.$$

Note that $F^* \cong F$.

The 2-category \mathcal{C}_A is \mathcal{J} -simple in the sense that any nonzero two-sided 2-ideal in \mathcal{C}_A contains the identity 2-morphisms on all indecomposable non-identity 1-morphisms, see [MM2, Subsection 6.2].

Denote by \mathcal{P} the full subcategory of \mathcal{A} consisting of projective objects. Then the defining action of \mathcal{C}_A on \mathcal{A} restricts to \mathcal{P} . We will denote the latter *defining additive* 2-representation of \mathcal{C}_A by \mathbf{D} .

Proposition 9. *For any $j = 1, \dots, n$ the 2-representations \mathbf{D} and $\mathbf{C}_{\mathcal{L}_j}$ are equivalent.*

Proof. It is easy to check that mapping the generator $P_{1\clubsuit}$ of \mathbb{P}_{\clubsuit} to the simple object in \mathcal{A} corresponding to j induces an equivalence from $\mathbf{C}_{\mathcal{L}_j}$ to \mathbf{D} . \square

5.2. Matrices in the Grothendieck group. Let \mathbf{M} be a finitary 2-representation of \mathcal{C}_A . For a 1-morphism G denote by $[G]$ the square matrix with non-negative integer coefficients whose rows and columns are indexed by isomorphism classes of indecomposable objects in $\mathbf{M}(\clubsuit)$ and the intersection of the row indexed by Y and the column indexed by X contains the multiplicity of Y as a direct summand of $G X$.

Consider the abelianization $\overline{\mathbf{M}}$ of \mathbf{M} . Then the isomorphism classes of simple objects in $\overline{\mathbf{M}}(\clubsuit)$ are in bijection with isomorphism classes of indecomposable objects in $\mathbf{M}(\clubsuit)$. For a 1-morphism G denote by $[[G]]$ the square matrix with non-negative integer coefficients whose rows and columns are indexed by isomorphism classes of simple objects in $\overline{\mathbf{M}}(\clubsuit)$ and the intersection of the row indexed by Y and the column indexed by X contains the composition multiplicity of Y in $G X$. The following generalizes [AM, Lemma 8].

Lemma 10. *We have $[[G^*]] = [G]^t$, where $_{}^t$ denotes the transpose of a matrix.*

Proof. For a projective P and a simple L in $\overline{\mathbf{M}}(\clubsuit)$ we have

$$\mathrm{Hom}_{\overline{\mathbf{M}}(\clubsuit)}(G P, L) \cong \mathrm{Hom}_{\overline{\mathbf{M}}(\clubsuit)}(P, G^* L).$$

The inclusion of $\mathbf{M}(\clubsuit)$ to $\overline{\mathbf{M}}(\clubsuit)$ given by $X \mapsto (0 \rightarrow X)$ is an equivalence between $\mathbf{M}(\clubsuit)$ and the category of projective objects in $\overline{\mathbf{M}}(\clubsuit)$. This implies the claim. \square

Lemma 11. *Consider the functor F from Subsection 5.1.*

- (i) *The matrix $[F]$ satisfies $[F]^2 = m[F]$.*
- (ii) *If \mathbf{M} is transitive, then all entries in $[F]$ are positive.*
- (iii) *If \mathbf{M} is transitive, then the rank of $[F]$ equals one.*

Proof. Claim (i) follows from (3). Claim (ii) is immediate from the definition of transitivity.

Claim (i) implies that $[F]$ is diagonalizable with eigenvalues 0 and m . By Theorem 1(ii), the eigenvalue m has multiplicity one. Claim (iii) follows. \square

5.3. Auxiliary lemmata.

Lemma 12. *Let \mathbf{M} be a simple transitive 2-representation of \mathcal{C}_A . Then for any $X \in \overline{\mathbf{M}}(\clubsuit)$ the object $F X$ is projective in $\overline{\mathbf{M}}(\clubsuit)$.*

Proof. Applying F to a minimal projective presentation $P_1 \xrightarrow{\alpha} P_0$ of $F X$ we get a projective presentation $F P_1 \xrightarrow{F(\alpha)} F P_0$ of $F^2 X \cong (F X)^{\oplus m}$.

Consider the split Grothendieck group of the category \mathcal{W} of projective objects in $\overline{\mathbf{M}}(\clubsuit)$. For $i = 0, 1$ let v_i be the vector recording the multiplicities of indecomposable projective objects in $F P_i$. Then, by minimality of the presentation $P_1 \xrightarrow{\alpha} P_0$, we have

$$(4) \quad [F] \cdot v_i = m v_i + w_i$$

for some non-negative vectors w_i . Note that $m v_i + w_i$ is a nonzero vector and belongs to the image of $[F]$. Therefore, by Lemma 11(iii), $m v_i + w_i$ is an eigenvector for $[F]$ with eigenvalue m . Hence $[F](m v_i + w_i) = m(m v_i + w_i)$. On the other hand,

$$[F](m v_i + w_i) = m[F]v_i + [F]w_i \stackrel{(4)}{=} m(m v_i + w_i) + [F]w_i.$$

Therefore $[F]w_i = 0$ and since w_i has only non-negative entries and all entries of $[F]$ are positive, we obtain $w_i = 0$.

It follows that $F P_1 \xrightarrow{F(\alpha)} F P_0$ is a minimal projective presentation of $F^2 X$, in particular, the morphism $F(\alpha)$ is contained in the radical of $\overline{\mathbf{M}}(\clubsuit)$.

The category \mathcal{W} carries the structure of a 2-representation of \mathcal{C}_A by restriction. This 2-representation is equivalent to \mathbf{M} (the natural inclusion of $\mathbf{M}(\clubsuit)$ into \mathcal{W} is the desired equivalence). In particular, the 2-representation of \mathcal{C}_A on \mathcal{W} is simple transitive. Let \mathcal{I} be the ideal of \mathcal{W} generated by $F(\alpha)$. This is contained in the radical of \mathcal{W} by the above and is F -stable by (3). Hence \mathcal{I} is \mathcal{C}_A -stable as it is stable under all indecomposable non-identity 1-morphisms. By simple transitivity, we thus get $\mathcal{I} = 0$, that is $\alpha = 0$. The claim follows. \square

Lemma 13. *Let B be a finite dimensional \mathbb{k} -algebra and G an exact endofunctor of $B\text{-mod}$. Assume that G sends each simple object of $B\text{-mod}$ to a projective object. Then G is a projective functor.*

Proof. Consider a short exact sequence of functors $K \hookrightarrow H \twoheadrightarrow G$ where H is a projective functor. This exists because any right exact functor is equivalent to tensoring with some bimodule and is hence a quotient of a projective functor. We assume that H is chosen minimally, that is such that the tops of H and G (viewed as bimodules) agree.

Applying $K \hookrightarrow H \twoheadrightarrow G$ to a short exact sequence $X \hookrightarrow Y \twoheadrightarrow Z$ in $B\text{-mod}$ we observe that $H X \twoheadrightarrow H Y \twoheadrightarrow H Z$ and hence the Snake Lemma yields the exact sequence $K X \hookrightarrow K Y \twoheadrightarrow K Z$. This implies that K is exact.

Applying $K \hookrightarrow H \twoheadrightarrow G$ to a simple object $L \in B\text{-mod}$ we obtain an exact sequence $KL \hookrightarrow HL \twoheadrightarrow GL$. By our choice of H , we have $HL = 0$ if and only if $GL = 0$. Furthermore, by assumptions on G we have $HL \cong GL$ whenever $GL \neq 0$. This implies $HL \cong GL$ for all L and hence $KL = 0$. By exactness of K we thus deduce $K = 0$ and hence $H \cong G$. \square

Lemma 14. *Let A , \mathcal{C}_A and F be as given in Subsection 5.1. Let further \mathbf{M} be a 2-representation of \mathcal{C}_A and $N \in \overline{\mathbf{M}}(\clubsuit)$ such that $FN \neq 0$. Then there is an algebra monomorphism from A to $\text{End}_{\overline{\mathbf{M}}(\clubsuit)}(FN)$.*

Proof. From the definitions we know that the 2-endomorphism algebra of F is isomorphic to $A \otimes_{\mathbb{k}} A^{\text{op}}$. We have a natural algebra monomorphism from A to $A \otimes_{\mathbb{k}} A^{\text{op}}$ given by $a \mapsto a \otimes 1$. Consider the evaluation homomorphism

$$\text{End}_{\mathcal{C}(\clubsuit, \clubsuit)}(F) \xrightarrow{\text{ev}_N} \text{End}_{\overline{\mathbf{M}}(\clubsuit)}(FN).$$

For a fixed left cell \mathcal{L} consider the corresponding cell 2-representation $\mathbf{C}_{\mathcal{L}}$ of \mathcal{C}_A . By [MM2, Proposition 21], there is a unique maximal left ideal in \mathcal{C}_A which does not contain any identity 2-morphisms for 1-morphisms in \mathcal{L} . Now, by [MM2, Subsection 6.5], this left ideal is the annihilator of the sum of all simple objects in $\overline{\mathbf{C}}_{\mathcal{L}}$. From Proposition 9 we know that $\mathbf{C}_{\mathcal{L}}$ is equivalent to the defining representation which implies that this maximal left ideal is, in fact, $A \otimes \text{rad } A^{\text{op}}$. Therefore the kernel of ev_N , which is a left ideal, must belong to $A \otimes \text{rad } A^{\text{op}}$.

This implies that the kernel of ev_N does not intersect the space $A \otimes 1$ and hence the induced composition $A \rightarrow \text{End}_{\mathcal{C}(\clubsuit, \clubsuit)}(F) \rightarrow \text{End}_{\overline{\mathbf{M}}(\clubsuit)}(FN)$ is injective. \square

5.4. Main result.

Theorem 15. *Let A be as given in Subsection 5.1. Then any simple transitive 2-representation of \mathcal{C}_A is equivalent to some cell 2-representation.*

Proof. Consider a simple transitive 2-representation \mathbf{M} of \mathcal{C}_A and its abelianization $\overline{\mathbf{M}}$. Let X_1, X_2, \dots, X_k be a complete and irredundant list of representatives of isomorphism classes of indecomposable objects in $\mathbf{M}(\clubsuit)$. Denote by B the endomorphism algebra of $\bigoplus_{i=1}^k X_i$. Note that $\overline{\mathbf{M}}(\clubsuit)$ is equivalent to $B^{\text{op-mod}}$. For $i = 1, 2, \dots, k$ we let L_i denote the simple quotient in $\overline{\mathbf{M}}(\clubsuit)$ of the indecomposable projective object $0 \rightarrow X_i$.

Recall the 1-morphism F defined in Subsection 5.1 and the corresponding matrix $[[F]]$ describing the action of F on the Grothendieck group of $\overline{\mathbf{M}}(\clubsuit)$ in the basis of simple modules. By Theorem 1(iv), there is a column,

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{pmatrix}$$

in $[[F]]$, say with index j , such that $v_1 + v_2 + \dots + v_k \leq m$. By Lemma 12 we have

$$F L_j \cong \bigoplus_{i=1}^k X_i^{\oplus l_i}$$

for some non-negative integers l_1, l_2, \dots, l_k . Transitivity of \mathbf{M} and (3) imply that all l_1, l_2, \dots, l_k are, in fact, positive integers. Denote by B' the endomorphism algebra of $F L_j$ which is Morita equivalent to B by the previous sentence. The vector $(l_1, l_2, \dots, l_k)^t$ is, by (3), an eigenvector of $[F]$. Moreover, by Lemma 10 we have $[F] = \llbracket F^* \rrbracket^t = \llbracket F \rrbracket^t$ where the latter equality follows from self-adjointness of F .

Lemma 14 provides an algebra embedding of A into B' and hence an embedding $A_A \hookrightarrow B'_A$ of A -modules. Since the algebra A (and hence also A^{op}) is self-injective, each indecomposable summand of A_A has simple socle. Therefore the embedding $A_A \hookrightarrow B'_A$ induces an embedding (of right A -modules) from A_A into

$$(5) \quad \bigoplus_{i=1}^k \text{Hom}_{\overline{\mathbf{M}}(\clubsuit)}(X_i, F L_j).$$

The dimension of the latter equals $v_1 + v_2 + \dots + v_k \leq m$, while $\dim_{\mathbb{k}} A = m$, therefore $v_1 + v_2 + \dots + v_k = m$ and by Corollary 2 all columns of $\llbracket F \rrbracket$ coincide. In particular, it follows that $l_1 = l_2 = \dots = l_k = l$ for some $l \in \mathbb{N}$ and thus B' is isomorphic to the algebra of $l \times l$ matrices with coefficients in B .

The algebra of B' -endomorphisms of (5) is isomorphic to B and embeds into the algebra of A -endomorphisms of (5) (the latter embedding is due to the fact that A is a subalgebra of B') which is equal to A by comparing dimensions. Therefore we have

$$B \hookrightarrow A \hookrightarrow B'.$$

Next we argue that $F L_s = (X_1 \oplus X_2 \oplus \dots \oplus X_k)^{\oplus l}$ for any s . The arguments above imply that $F L_s = (X_1 \oplus X_2 \oplus \dots \oplus X_k)^{\oplus l_s}$ for some positive integer l_s . Now $l = l_s$ since all columns of $\llbracket F \rrbracket$ are equal.

As $F L_j = (X_1 \oplus X_2 \oplus \dots \oplus X_k)^{\oplus l}$, it follows that $\dim_{\mathbb{k}} B' = lm$ and therefore $\dim_{\mathbb{k}} B = \frac{m}{l}$. Set $\Theta := \overline{\mathbf{M}}(F)$. Lemma 13 implies that Θ is a projective functor which sends each simple to $(X_1 \oplus X_2 \oplus \dots \oplus X_k)^{\oplus l}$. The dimension of the endomorphism algebra of Θ thus equals $l \cdot l \cdot \frac{m}{l} \cdot \frac{m}{l} = m^2$. Note that \mathcal{J} -simplicity of \mathcal{C}_A gives us a natural inclusion of the algebra $\text{End}_{\mathcal{C}(\clubsuit, \clubsuit)}(F) \cong A \otimes A^{\text{op}}$ of 2-endomorphisms of F into the endomorphism algebra of Θ in the category of right exact endofunctors of $\overline{\mathbf{M}}(\clubsuit)$. As both these algebras have dimension m^2 , this natural inclusion is, in fact, an isomorphism.

Therefore $B \cong A \cong B'$ and thus $\overline{\mathbf{M}}$ is equivalent to the defining 2-representation of \mathcal{C}_A . Now the proof is completed by applying Proposition 9. \square

5.5. Generalizations.

Remark 16. Theorem 15 generalizes verbatim and with the same proof to the case where A is a basic self-injective finite dimensional \mathbb{k} -algebra (not necessarily connected). The technical difficulty in this case is that, in order to be consistent with the requirement for $\mathbb{1}_i$ to be indecomposable, one has to consider a 2-category with several objects indexed by connected components of A .

Remark 17. Theorem 15 generalizes verbatim to 2-subcategories of \mathcal{C}_A described in [MM3, Subsection 4.5]. These 2-subcategories exhaust all “simple” 2-categories of certain type, see [MM3, Theorem 13] and Subsection 6.1 below for details. The only difference between those 2-subcategories and \mathcal{C}_A is that the former may contain fewer 2-endomorphisms of the identity 1-morphisms. We did not use 2-endomorphisms of identity 1-morphisms in the above proof.

6. TRANSITIVE 2-REPRESENTATIONS FOR SOME GENERAL FIAT 2-CATEGORIES

6.1. Strong regularity and a numerical condition. Let \mathcal{C} be a fiat 2-category and \mathcal{J} a two-sided cell in \mathcal{C} . We say that \mathcal{J} is *strongly regular*, see [MM1, Subsection 4.8], provided that

- different right (left) cells in \mathcal{J} are not comparable with respect to the right (left) preorder;
- the intersection of a left and a right cell in \mathcal{J} consists of exactly one isomorphism class of indecomposable 1-morphisms.

For example, the 2-category \mathcal{C}_A from Subsection 5.1 is strongly regular.

If \mathcal{J} is strongly regular, we have a well-defined function sending $F \in \mathcal{J}$ to the number of indecomposable summands in $F^* \circ F$ which belong to \mathcal{J} . We will say that \mathcal{J} satisfies the *numerical condition* provided that this function is constant on right cells. Again, it is easy to check that the 2-category \mathcal{C}_A from Subsection 5.1 satisfies the numerical condition, see [MM1, Subsection 7.3].

Another example of a 2-category in which each two-sided cell is strongly regular and satisfies the numerical condition is the 2-category \mathcal{S}_n of Soergel bimodules for the symmetric group S_n , see [MM1, Subsection 7.1] and [MM2, Example 3] for details.

6.2. Another generalization of the main result.

Theorem 18. *Let \mathcal{C} be a fiat 2-category such that all two-sided cells in \mathcal{C} are strongly regular and satisfy the numerical condition. Then any simple transitive 2-representation of \mathcal{C} is equivalent to a cell 2-representation.*

Proof. Let \mathbf{M} be a simple transitive 2-representation of \mathcal{C} . First of all, we claim that there is a unique maximal two-sided cell \mathcal{J} which does not annihilate \mathbf{M} . Indeed, assume that we have two maximal two-sided cells \mathcal{J}_i for $i = 1, 2$ with this property. Then for any $F_i \in \mathcal{J}_i$, $i = 1, 2$, we have $\mathbf{M}(F_1) \circ \mathbf{M}(F_2) = 0$ and $\mathbf{M}(F_2) \circ \mathbf{M}(F_1) = 0$ whenever the expression makes sense. Therefore the additive closure of objects in all $\mathbf{M}(\mathcal{J}_i)$ which may be obtained by applying 1-morphisms from \mathcal{J}_1 is, on the one hand, a 2-subrepresentation of \mathbf{M} (by maximality of \mathcal{J}_1) and, on the other hand, annihilated by all 1-morphisms from \mathcal{J}_2 . Due to transitivity of \mathbf{M} , we obtain that \mathcal{J}_2 annihilates \mathbf{M} , a contradiction.

Now denote by \mathcal{J} the maximal two-sided cell of \mathcal{C} which does not annihilate \mathbf{M} . Without loss of generality we may assume that \mathcal{J} is the unique maximal two-sided cell in \mathcal{C} and that \mathbf{M} is 2-faithful in the sense that it does not annihilate any 2-morphisms. Indeed, we may replace \mathcal{C} by its quotient modulo the kernel of \mathbf{M} which does not change the structure of the surviving cells.

Denote by $\mathcal{C}_{\mathcal{J}}$ the 2-full 2-subcategory of \mathcal{C} formed by all 1-morphisms in \mathcal{J} together with their respective identity 1-morphisms. By restriction, \mathbf{M} becomes a 2-representation $\mathbf{M}_{\mathcal{J}}$ of $\mathcal{C}_{\mathcal{J}}$. As the additive closure of 1-morphisms in \mathcal{J} is stable with respect to left multiplication by 1-morphisms in \mathcal{C} , it follows that \mathbf{M} is a transitive 2-representation of $\mathcal{C}_{\mathcal{J}}$.

We claim that $\mathbf{M}_{\mathcal{J}}$ is simple transitive. Indeed, assume that \mathbf{J} is an ideal of \mathbf{M} stable with respect to the action of $\mathcal{C}_{\mathcal{J}}$. Assume that it is nonzero and take any nonzero morphism α in it. As \mathbf{M} is a simple transitive 2-representation of \mathcal{C} , there exists a 1-morphism G in \mathcal{C} such that $G(\alpha)$ has an invertible nonzero direct summand. Applying 1-morphisms from $\mathcal{C}_{\mathcal{J}}$ we, on the one hand, will map such an invertible direct summand to another invertible morphism (and since \mathbf{M} is transitive there is a 1-morphism F in $\mathcal{C}_{\mathcal{J}}$ which does not annihilate this invertible direct summand). On the other hand, $F \circ G$ is in \mathcal{J} and hence application of it to α cannot produce any invertible direct summands, a contradiction. Therefore \mathbf{J} is zero.

By Theorem 15, Remark 17 and [MM3, Theorem 13], $\mathbf{M}_{\mathcal{J}}$ is equivalent to a cell 2-representation $\mathbf{C}_{\mathcal{L}}^{\mathcal{J}}$ of $\mathcal{C}_{\mathcal{J}}$ where \mathcal{L} is a left cell in \mathcal{J} . By [MM1, Theorem 43] any choice of \mathcal{L} yields an equivalent 2-representation. Set $\mathbf{i} = \mathbf{i}_{\mathcal{L}}$ and let L be a simple object in $\overline{\mathbf{C}}_{\mathcal{L}}^{\mathcal{J}}(\mathbf{i})$ which is not annihilated by 1-morphisms in \mathcal{L} . Then we can consider L as an object in $\overline{\mathbf{M}}(\mathbf{i})$.

Sending $P_{\mathbf{i}_1}$ to L gives a 2-natural transformation Φ from the 2-representation $\mathbb{P}_{\mathbf{i}_1}$ of \mathcal{C} to $\overline{\mathbf{M}}$. In the notation of Subsection 3.3, the image of $\mathbf{N}(\mathbf{j})$ for $\mathbf{j} \in \mathcal{C}$ under Φ is inside the category of projective objects in $\overline{\mathbf{M}}(\mathbf{j})$ and contains at least one representative in each isomorphism class of indecomposable objects, see [MM1, Subsection 4.5]. We also have that \mathbf{I} (see Subsection 3.3) annihilates L by construction. It follows that the 2-representation \mathbf{K} of \mathcal{C} on projective objects in the categories $\overline{\mathbf{M}}(\mathbf{j})$ (for $\mathbf{j} \in \mathcal{C}$) is equivalent to the cell 2-representation $\mathbf{C}_{\mathcal{L}}$ of \mathcal{C} . As \mathbf{K} is equivalent to \mathbf{M} by [MM2, Theorem 11], we deduce that \mathbf{M} is equivalent to $\mathbf{C}_{\mathcal{L}}$. This completes the proof. \square

7. EXAMPLES

7.1. A non weakly fiat 2-category \mathcal{C}_A . In this subsection we give an example of a non weakly fiat 2-category \mathcal{C}_A for which Theorem 15 generalizes to the class of exact simple transitive 2-representations. Taking into account the example considered in Subsection 3.4, the present example is somewhat surprising.

For $A = \mathbb{k}[x, y]/(x^2, y^2, xy)$ consider the 2-category \mathcal{C}_A as defined in Subsection 5.1. Note that A is local but not self-injective which implies that \mathcal{C}_A is not weakly fiat. Let F be an indecomposable 1-morphism in \mathcal{C}_A which is not isomorphic to the identity 1-morphism. The defining 2-representation of \mathcal{C}_A is easily seen to be equivalent to the cell 2-representation $\mathbf{C}_{\mathcal{L}}$ for $\mathcal{L} = \{F\}$.

Proposition 19. *For $A = \mathbb{k}[x, y]/(x^2, y^2, xy)$, any exact simple transitive 2-representation of \mathcal{C}_A is equivalent to a cell 2-representation.*

Proof. Let \mathbf{M} be an exact simple transitive 2-representation of \mathcal{C}_A . Without loss of generality we may assume $\mathbf{M}(F) \neq 0$. Then $F \circ F \cong F^{\oplus 3}$ and hence $[[F]]^2 = 3[[F]]$ by exactness of $\mathbf{M}(F)$. Using Theorem 1 it is easy to check that $[[F]]$ is equal to one of the following matrices:

$$M_1 := \begin{pmatrix} 3 \end{pmatrix}, \quad M_2 := \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \quad M_3 := \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \quad M_4 := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Let B and B' be as in the proof of Theorem 15. Note that both Lemma 12 and Lemma 14 are still applicable in our situation. Despite of the fact that \mathcal{C}_A is not weakly fiat, it is still \mathcal{J} -simple, where $\mathcal{J} = \{F\}$.

If $\llbracket F \rrbracket = M_4$, then $B \cong \mathbb{k}^{\oplus 3}$ and $\mathbf{M}(F)$ is the direct sum of nine copies of the identity functors (between the three different copies of $\mathbb{k}\text{-mod}$). The endomorphism algebra of $\mathbf{M}(F)$ has thus dimension nine and is clearly not isomorphic to $A \otimes_{\mathbb{k}} A^{\text{op}}$. Hence this case is not possible.

If $\llbracket F \rrbracket = M_3$, then $B = B' \cong \mathbb{k}^{\oplus 2}$ and the algebra A does not inject into B' . This contradicts Lemma 14 and hence this case is not possible either.

If $\llbracket F \rrbracket = M_2$, then either $B = B'$ is a 3-dimensional algebra which is not local or $B \cong \mathbb{k}^{\oplus 2}$ and $B' \cong \mathbb{k} \oplus \text{Mat}_{2 \times 2}(\mathbb{k})$. In the first case we again get a contradiction to Lemma 14. In the second case the endomorphism algebra of $\mathbf{M}(F)$ has dimension ten and two direct summands isomorphic to \mathbb{k} , say this endomorphism algebra is $Q \oplus \mathbb{k} \oplus \mathbb{k}$. If the local algebra $A \otimes_{\mathbb{k}} A^{\text{op}}$ were to inject into the endomorphism algebra of $\mathbf{M}(F)$, the algebra $A \otimes_{\mathbb{k}} A^{\text{op}}$ would also inject into Q which has strictly smaller dimension, a contradiction. Hence this case is not possible.

If $\llbracket F \rrbracket = M_1$, then either $B \cong \mathbb{k}$ and $B' = \text{Mat}_{3 \times 3}(\mathbb{k})$ or $B = B'$ has dimension 3. In the former case the endomorphism algebra of $\mathbf{M}(F)$ has dimension nine and is not local, implying a contradiction similarly to the case $\llbracket F \rrbracket = M_4$. In the latter case we again use Lemma 14 to get $B = B' \cong A$ and then we readily deduce that \mathbf{M} is equivalent to the cell 2-representation. \square

7.2. Categorification of finite dimensional 2-Lie algebras. Let \mathfrak{g} denote a simple finite dimensional complex Lie algebra. We fix a triangular decomposition $\mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ of \mathfrak{g} . For any \mathfrak{h} -weight λ denote by $L(\lambda)$ the corresponding simple highest weight module with highest weight λ . Let \leq denote the natural partial order on \mathfrak{h} -weights.

Let \mathcal{U} be the 2-category categorifying the idempotent version \hat{U} of the universal enveloping algebra of \mathfrak{g} as defined in [We, Definition 2.4] (the origins of this 2-category are in [CL], see also [KL, Ro1] for other variations). The categorification statement is justified by [We, Theorem B.2]. For each dominant integral \mathfrak{h} -weight λ there is a 2-representation of \mathcal{U} given by a functorial action on the direct sum (over n) of categories of projective modules over the cyclotomic quiver Hecke algebras (KLR algebras) R_n^λ associated with \mathfrak{g} (see [We, Theorem 3.17] for \mathcal{U} and also [KK, Ka] for a similar statement related to Rouquier's 2-Kac-Moody algebras). This 2-representation categorifies $L(\lambda)$. We note the following properties of this 2-representation:

- As $L(\lambda)$ is finite dimensional, only finitely many of the algebras R_n^λ are non-zero.
- As $L(\lambda)$ is finite dimensional, sufficiently high powers of the generators annihilate our 2-representation. Hence, the commutation relations in \mathfrak{g} imply that only finitely many indecomposable 1-morphisms from \mathcal{U} act as non-zero functors in this 2-representation.
- Each R_n^λ is finite dimensional and all involved functors are exact.

- Each 1-morphism in \mathcal{U} acts as an exact functor and hence can be realized as tensoring with a finite-dimensional bimodule. This implies that the spaces of two morphisms in this 2-representation are finite dimensional.
- Each 1-morphism in \mathcal{U} has a biadjoint which is again a functor representing the action of some 1-morphism in \mathcal{U} .
- The endomorphism algebra of each identity 1-morphism in \mathcal{U} is positively graded by the non-degeneracy part of [We, Theorem B.2] and isomorphic to a polynomial ring ([We, Proposition 3.31]). In particular, each finite dimensional graded quotient of this algebra is local.

Let \mathcal{I}_λ be the kernel of this 2-representation and set $\mathcal{U}_\lambda := \mathcal{U}/\mathcal{I}_\lambda$. Then the above implies that \mathcal{U}_λ is a fiat 2-category. Note that \mathcal{I}_λ is, in general, not generated by 2-morphisms of the form id_F , where F is some 1-morphism, but it additionally contains some of the 2-morphisms between 1-morphisms which are not in \mathcal{I}_λ , see [MM2, Remark 31].

Consider a finite set $\lambda := \{\lambda_1, \lambda_2, \dots, \lambda_k\}$ of dominant integral \mathfrak{h} -weights such that $\lambda_i \not\leq \lambda_j$ for all $i \neq j$ and denote by $\bar{\lambda}$ the set of all dominant integral weights μ such that $\mu \leq \lambda_i$ for some i . Note that $\bar{\lambda}$ is a finite set. Define

$$\mathcal{U}_\lambda := \mathcal{U}/(\mathcal{I}_{\lambda_1} \cap \mathcal{I}_{\lambda_2} \cap \dots \cap \mathcal{I}_{\lambda_k}),$$

which is again a fiat 2-category.

Remark 20. Let \mathcal{L} be the left cell in \mathcal{U}_λ containing the indecomposable 1-morphism $\mathbb{1}_{\lambda_l}$ for $l \in \{1, 2, \dots, k\}$. As $\mathbb{1}_{\lambda_l}$ is a genuine idempotent and is, obviously, the unique element in the intersection of its left and right cells, the radical of its endomorphism ring is contained in the ideal \mathbf{I} from Subsection 3.3 used to define the corresponding cell 2-representation $\mathbf{C}_\mathcal{L}$. Consequently, the image of $\mathbb{1}_{\lambda_l}$ in the abelianized cell 2-representation is both simple and projective (this corresponds to a projective module over $R_0^\lambda \cong \mathbb{C}$). Moreover, the functor $\mathbf{C}_\mathcal{L}(\mathbb{1}_{\lambda_l})$ is just the identity functor on the category of complex vector spaces, in particular, its endomorphism ring consists only of scalars. Note that our construction of $\mathbf{C}_\mathcal{L}$ differs, in particular, from the construction of the universal categorification of $L(\lambda)$ in [Ro1, Subsection 5.1.2]. In the latter case the endomorphism of $\mathbb{1}_{\lambda_l}$ is much bigger in general.

Theorem 21. *For any λ as above every two-sided cell in the 2-category \mathcal{U}_λ is strongly regular and satisfies the numerical condition.*

Proof. For $l \in \{1, 2, \dots, k\}$ consider the two-sided cell \mathcal{J} of \mathcal{U}_λ containing $\mathbb{1}_{\lambda_l}$. Then, factoring out the maximal 2-ideal in \mathcal{U}_λ which contains $\text{id}_{\mathbb{1}_{\lambda_l}}$ and does not contain the identity 2-morphism for any 1-morphism outside \mathcal{J} (note that such an ideal does not have to be generated by 2-morphisms of the form id_F , where F is some 1-morphism), we obtain the 2-category \mathcal{U}_μ where μ is uniquely defined via $\bar{\mu} := \bar{\lambda} \setminus \{\lambda_l\}$, cf. [DG, Section 9]. Therefore it is enough to prove that \mathcal{J} is strongly regular and satisfies the numerical condition.

Let \mathcal{L} denote the left cell of $\mathbb{1}_{\lambda_l}$. Let further L be an indecomposable object in $R_0^\lambda\text{-proj}$. Note that $R_0^\lambda \cong \mathbb{C}$. As L corresponds to the highest weight vector in $L(\lambda)$, all 1-morphisms which do not annihilate L must correspond to the $U(\mathfrak{n}_-)$ part of U . This means that \mathcal{L} consists of direct summands of powers of the negative generators of \mathcal{U} . Then, from [We, Theorem 3.17] in combination with [Ro1, Theorem 5.7]

and [VV, Theorem 4.4], it follows that mapping an indecomposable 1-morphism $F \in \mathcal{L}$ to FL induces a bijection between \mathcal{L} and the set of isomorphism classes of indecomposable objects in

$$\bigoplus_{n \geq 0} R_n^\lambda\text{-proj.}$$

Set

$$A := \bigoplus_{n \geq 0} R_n^\lambda \quad \text{and} \quad B := \bigoplus_{n \geq 1} R_n^\lambda.$$

For any $M \in B\text{-proj}$ we have $\mathbb{1}_{\lambda_l} M = 0$ and therefore $FM = 0$ for any $F \in \mathcal{L}$. Consider the abelian 2-representation $\overline{\mathbf{C}}_{\lambda_l}$.

Since \mathcal{U}_λ is fiat, Lemma 13 implies that $\overline{\mathbf{C}}_{\lambda_l}(F)$ is an indecomposable projective functor from $\mathbb{C}\text{-mod}$ to $A\text{-mod}$. Consequently, for any $G \in \mathcal{L}$ the functor $\overline{\mathbf{C}}_{\lambda_l}(F \circ G^*)$ is indecomposable. We claim that this implies that $F \circ G^*$ is indecomposable. Indeed, if $F \circ G^* \cong X \oplus Y$, then without loss of generality we may assume $\overline{\mathbf{C}}_{\lambda_l}(Y) = 0$. Since \mathcal{J} is a maximal two-sided cell, we have $Y \in \mathcal{J}$ and hence $\overline{\mathbf{C}}_{\lambda_l}(Y) \neq 0$, a contradiction.

The previous paragraph shows that the set $\{F \circ G^*\}$, where $F, G \in \mathcal{L}$, consists of indecomposable 1-morphisms and hence coincides with \mathcal{J} . In particular $|\mathcal{J}| = |\mathcal{L}|^2$. It is now obvious that the left cells in \mathcal{J} are obtained fixing G and the right cells in \mathcal{J} are obtained fixing F . Therefore \mathcal{J} is strongly regular. To check the numerical condition we note that $\overline{\mathbf{C}}_{\lambda_l}$ realizes elements of \mathcal{J} as tensoring with indecomposable projective A - A -bimodules, so the numerical condition follows from [MM1, Subsection 7.3]. \square

7.3. Soergel bimodules in type B_2 . Consider the 2-category \mathcal{S} of Soergel bimodules for a Lie algebra of type B_2 , see [MM1, Section 7.1] and [MM2, Example 20]. We denote by \clubsuit the (unique) object in \mathcal{S} . The Weyl group in this case is given by

$$W = \{e, s, t, st, ts, sts, tst, stst = tsts\},$$

where $s^2 = t^2 = e$, and is isomorphic to the dihedral group D_4 . The group D_4 has five simple modules over \mathbb{C} : the one-dimensional simple modules $V_{\varepsilon, \delta}$, for $\varepsilon, \delta \in \{\pm 1\}$, where s acts via ε and t acts via δ ; and the 2-dimensional simple module V_2 (the defining geometric representation). For an additive category \mathcal{A} we denote by $K_0(\mathcal{A})$ the split Grothendieck group of \mathcal{A} . Our aim in this section is to apply previous results in order to prove the following statement which describes simple W -modules admitting a finitary categorification.

Proposition 22. *Let \mathbf{M} be a finitary 2-representation of \mathcal{S} . Assume that the induced action of the algebra $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{S}(\clubsuit, \clubsuit))$ on the vector space $\mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathbf{M}(\clubsuit))$ gives a simple W -module V . Then $V \cong V_{1,1}$ or $V \cong V_{-1,-1}$.*

Proof. We have three two-sided cells

$$\mathcal{J}_1 = \mathcal{L}_1 = \{e\}, \quad \mathcal{J}_2 = \{s, t, st, ts, sts, tst\}, \quad \mathcal{J}_3 = \mathcal{L}_3 = \{stst\}$$

and J_2 splits into two left cells

$$\mathcal{L}_2^{(1)} = \{s, st, sts\} \quad \text{and} \quad \mathcal{L}_2^{(2)} = \{t, ts, tst\}.$$

Right cells are obtained using the map $w \mapsto w^{-1}$.

It is easy to check that the cell 2-representations $\mathbf{C}_{\mathcal{L}_1}$ and $\mathbf{C}_{\mathcal{L}_3}$ categorify $V_{1,1}$ and $V_{-1,-1}$, respectively.

We identify indecomposable Soergel bimodules θ_w for $w \in W$ with the corresponding elements

$$\begin{aligned}\theta_e &= e, & \theta_s &= e + s, & \theta_t &= e + t, & \theta_{st} &= e + t + s + st, & \theta_{ts} &= e + t + s + ts, \\ \theta_{sts} &= e + t + s + ts + st + sts, & \theta_{tst} &= e + t + s + ts + st + tst, \\ \theta_{stst} &= e + t + s + ts + st + tst + sts + stst\end{aligned}$$

in the Kazhdan-Lusztig basis for $\mathbb{Z}[W]$.

Note that the element θ_s annihilates $V_{-1,1}$ while θ_t does not annihilate $V_{-1,1}$. If we had a 2-representation \mathbf{M} decategorifying to $V_{-1,1}$, then $\mathbf{M}(\theta_s) = 0$ while $\mathbf{M}(\theta_t) \neq 0$ which is impossible as θ_s and θ_t belong to the same two-sided cell. Therefore $V \not\cong V_{-1,1}$ and, by symmetry, $V \not\cong V_{1,-1}$. (This argument came up in discussion with Catharina Stroppel.)

It is left to show that $V \not\cong V_2$. Note that θ_{stst} annihilates V_2 . Assume that \mathbf{M} is a 2-representation of \mathcal{S} decategorifying to V_2 and consider $\overline{\mathbf{M}}$. Set $\Theta := \sum_{w \in J_2} \theta_w$. Direct computation shows that

$$(\theta_{st} + \theta_{ts})^2 = 2\Theta \pmod{J_3}, \quad \Theta^2 = 10\Theta + 4(\theta_{st} + \theta_{ts}) \pmod{J_3}.$$

This implies that the matrix $X := \llbracket \theta_{st} + \theta_{ts} \rrbracket$ satisfies the polynomial equation $X^4 - 20X^2 - 16X = 0$. Consequently, X is diagonalizable with eigenvalues in $\{0, -4, 2(1 \pm \sqrt{2})\}$. Clearly, X is not the zero matrix. As all entries of X are non-negative, the trace of X is non-negative which implies that the eigenvalues of X are $2(1 \pm \sqrt{2})$, each with multiplicity one. Thus the trace of X is 4 and the determinant is -4 , leaving

$$\begin{aligned}& \begin{pmatrix} 4 & 4 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 4 & 2 \\ 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 4 & 1 \\ 4 & 0 \end{pmatrix}, \quad \begin{pmatrix} 3 & 7 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 \\ 7 & 1 \end{pmatrix}, \\ & \begin{pmatrix} 2 & 8 \\ 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 4 \\ 2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 2 \\ 4 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ 8 & 2 \end{pmatrix}\end{aligned}$$

as possibilities (up to reordering of the basis).

We have $\theta_s^2 \cong 2\theta_s$ and $\theta_t^2 \cong 2\theta_t$, which implies that both $\llbracket \theta_s \rrbracket$ and $\llbracket \theta_t \rrbracket$ satisfy the polynomial equation $x^2 - 2x = 0$. Similarly to the above, this leads to the list of candidates for $\llbracket \theta_s \rrbracket$ and $\llbracket \theta_t \rrbracket$ being given by

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & a \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ a & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

where $a \in \{0, 1, 2, \dots\}$. Note that $\theta_{st} = \theta_s \theta_t$ and $\theta_{ts} = \theta_t \theta_s$. Hence, the equation

$$\llbracket \theta_{st} + \theta_{ts} \rrbracket = \llbracket \theta_s \rrbracket \llbracket \theta_t \rrbracket + \llbracket \theta_t \rrbracket \llbracket \theta_s \rrbracket$$

reduces the choice to

$$(6) \quad \llbracket \theta_s \rrbracket = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \llbracket \theta_t \rrbracket = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

or

$$(7) \quad \llbracket \theta_s \rrbracket = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}, \quad \llbracket \theta_t \rrbracket = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}$$

or vice versa.

In case of (6), we may restrict \mathbf{M} to the 2-subcategory \mathcal{T} of \mathcal{S} generated by θ_e and θ_s and adjunction morphisms between them. This 2-category clearly satisfies all hypotheses of Theorem 18. Note that θ_s is self-adjoint, hence Lemma 10 implies that this restricted 2-representation is transitive. Let \mathbf{N} be its simple transitive

quotient. Then \mathbf{N} gives rise to a simple transitive 2-representation of \mathcal{T} in which θ_s has the matrix described by (6). This, however, contradicts Theorem 18.

In case of (7), consider $\overline{\mathbf{M}}$ and let L_1 and L_2 denote the simple objects in $\overline{\mathbf{M}}(\clubsuit)$. Note that, by adjunction, a simple object L can appear in the top or in the socle of some $\theta_x N$, for $x \in \{s, t\}$, only if $\theta_x L \neq 0$. Therefore $\theta_s L_1$ cannot have L_2 in top or socle and hence is uniserial of Loewy length three with simple top and simple socle isomorphic to L_1 . Similarly, the module $\theta_t L_2$ has simple top and simple socle isomorphic to L_2 . Let M denote the homology in the middle term of

$$0 \rightarrow L_2 \rightarrow \theta_t L_2 \rightarrow L_2 \rightarrow 0.$$

Then M has length two with both simple subquotients isomorphic to L_1 .

Assume $M \cong L_1 \oplus L_1$. Let N be a non-split extension of length two with top L_1 and socle L_2 . Then, since $\theta_s L_2 = 0$ and θ_s is exact, by adjunction we have

$$\dim \operatorname{Hom}(\theta_s L_1, N) = \dim \operatorname{Hom}(L_1, \theta_s L_1) = 1$$

and thus N is a quotient of $\theta_s L_1$. This implies $\dim \operatorname{Ext}^1(L_1, L_2) = 1$. At the same time, consider $\operatorname{Rad}(\theta_t L_2)$. By the above, this has simple socle L_2 , the quotient over which is M . Hence $\dim \operatorname{Ext}^1(L_1, L_2) \geq 2$, a contradiction.

Assume that M is indecomposable. Then, by adjunction,

$$1 = \dim \operatorname{Hom}(M, \theta_s L_1) = \dim \operatorname{Hom}(\theta_s M, L_1).$$

As only L_1 can be in the top of M , the module $\theta_s M$ has simple top and hence is indecomposable. Consequently, $\theta_s \theta_t \theta_s L_1 \cong \theta_s M$ is indecomposable. However, $\theta_s \theta_t \theta_s = \theta_{sts} \oplus \theta_s$ and thus $\theta_s M$ has $\theta_s L_1$ as a direct summand. As dimensions of $\theta_s M$ and $\theta_s L_1$ are different, this is a contradiction. The proof is complete. \square

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